Chapter 4

4.1 Write the following recurrence equation for the following algorithms :

a) n^{th} power of a variable : T(n) = 1 + T(n-1)

b) Pentagonal numbers : T(1)=1, T(n) = T(n-1)+3n-2 , $n \geq 2$

c) Quick Sort : T(n) = 2T

d) Selection Sort : T(n) = T(n-1)+1

e) Lucas numbers : T(n) = T(n-1)+T(n-2)

T(0) = 1

4.2 Formulate and Solve the following recurrences

a) $t_n=t_{n-1} + 3$, $t_0 = 0$

$\therefore$ when n=1 $t_1 = t_0 + 3 = 3$

$t_2 = t_1 + 3 = 6$

$t_3 = t_2 + 3 = 9$

$\therefore$ The sequence is 0, 3, 6, 9, $\cdots$

Solution is $t_n = 3n$, for $n = 0,1,2,\cdots$

b) $t_n=2t_{n-1}+1$, $t_0=0$

$t_1 = 2t_0 + 1 = 1$

$t_2 = 2t_1 + 1 = 3$

$t_3 = 2t_2 + 1 = 7$

$t_4 = 2t_3 + 1 = 15$

So the sequence is 0,1,3,7,1.5,$\cdots$

So guessed solution is $2\text{n}+1$, for $\text{n} = 0,1,2,\ldots$

c) $t_n = 3t_{n-1}+1$, $t_0=0$

$t_1 = 3t_0 + 1 = 1$

$t_2 = 3t_1 + 1 = 5$

$t_3 = 3t_2 + 1 = 10$

$t_4 = 3t_3 + 1 = 58$

So the sequence is $0,1,5,10,58,\cdots$

d) $t_n = 2t_{n-1}+\text{n}^3$, $t_0 = 0$

$t_1 = 2t_0 + 1^3 = 1$

$t_2 = 2t_1 + 2^3 = 10$

$t_3 = 2t_2 + 3^3 = 20 + 37 = 47$

$t_4 = 2t_3 + 4^3 = 2\times47 + 64 = 158$

So the sequence is 0,1,10,47,158

e) $t_n = 5t_{n-1}+n^2$, $t_0 = 0$

$t_1 = 5t_0 + 1^2 = 1$

$t_2 = 5t_1 + 1^2 = 9$

$t_3 = 5t_2 + 1^2 = 45 + 9 = 54$

$t_4 = 5t_3 + 1^2 = 5\times54 + 16 = 286$

So the sequence is 0,1,9,54,286

4.3 Solve the following 1st order equation using substitution method.

a) t_n-4t_n-1 = 0

$t_\text{n} = 4t_n\text{-}1$

$= 4[4(t_{n-2})] = 4^2t_{n-2}$

$= 4[4(4t_{n-3})] = 4^3t_{n-3}$

$= 4^4t_{n-4}$

$\therefore$ In general

$\boxed{t_n= 4^{n-i}\times t_{n-1}}$

when$I = (n–1)$, the solution would become like this

$t_n = 4^{n-(n-1)}\times t_{n-1}$

$\boxed{t_n = 4\times \text{t}_{n-1}}$

4.4 Solve the following recurrence equations using the difference method :

a) $t_n - t_{n-1} = 7$

$t_0 = 1$

$t_n - t_{n-1} = 7$

$t_{n-1} - t_{n-2} = 7$

$t_{n-2} - t_{n-3} = 7$

$\vdots$

$t_2 - t_{n-1} = 7$

There are (n-1) equations

$\therefore t_n - t_1 = 7(n\text{-}1)$

$t_n = t_1 + 7(n\text{-}1)$

$= 1 + 7n - 7 = 7n - 6$

b) $t_n - t_{n-1} = 3$

Based on the previous problem, one can find

$t_n = t_1 + 3(n\text{-}1)$

$= 1 + 3n\text{-}3$

$= 3n\text{-}2$

4.5 Solve the following problems using recurrence tree.

a) T(n) = 2T(n-1) + 1 and compare this with T(n) = 2T(n-1) + n

Hint: At every stage 1 is divided into two leaves and this continues till it becomes 1. $\sum_{i=1}^{n} 2^i = 2^n - 1 = O(2^n)$

b) $T(n) = T\left(\frac{n}{2}\right) + 1$ Compare this with the tree $T\left(\frac{n}{2}\right) + n$

Hint: At every stage 1 is divided to two leaves as n/2 and this continues till it becomes 1. This is O(log n)

c) $T(n) = 4T\left(\frac{n}{2}\right) + n^2$

Hint: At every stage the problem is divided into 4 sub-problems and every time n is divided by 2 and each call requires n^2. Therefore, the algorithm is $\theta(n^2 \log n)$.

d) $T(n) = 3T\left(\frac{n}{2}\right) + n$

Hint: At every stage the problem is divided into 3 sub-problems and every time n is divided by 2 and each call requires n^2. Therefore, the algorithm is $\theta(n^2 \log n)$.

$$\sum_{i=0}^{\log n-1} \left(\frac{3}{2}\right)^i = n^{\log 3} - 1$$

4.6 Solve the following second order equation

a) $t_n - 7t_{n-1} + 12t_{n-2}$ for $n = 0$

$t_0 = 0$

$t_1 = 1$

***Solution*:**

$r^2 - 7r + 12 = 0$

$$= \frac{+7 \pm \sqrt{49 - 4(1)(12)}}{2(1)}$$

$$= \frac{+7 \pm \sqrt{49-48}}{2} = \frac{+7+1}{2}, \frac{+7-1}{2}$$

$$= +3, +4$$

$\therefore t_n = c_1 3^n + c_2 4^n$

When $n = 0$, $t_0 = c_1 + c_2$

When $n = 1$ $t_1 = 3c_1 + 4c_2$

Solving this one get

$c_1 + c_2 = 0$

$3c_1 + 4c_2 = 1$

eqn1 $\times$ 3

$3c_1 + 3c_2 = 0$

$3c_1 + 4c_2 = 1$

$c_2 = 1$

$\therefore c_2 = 1, \;\; c_1 = -1$

Substituting this, one gets the equation

$t_n = (-1)3^n + 4^n$

b) $t_n - 3t_{n-1} - 4t_{n-2} = 0$, $t_0 = 0$, $t_1 = 1$

Solution:

$r^3 - 3r - 4 = 0$

$r = 4, -1$

$\therefore t_n \; = \; c_1 4^n + c_2(-1)^n$

Substituting the initial conditions, one get

$t_0 \; = \; c_1 4^n + c_2(-1)^n$

$t_1 \; = \; 4c_1 - c_2 = 1$

Solving one get $c_1 = \frac{1}{5}, c_2 = \frac{-1}{5}$

$\therefore t(n) = \frac{1}{5}(4)^n - \frac{1}{5}(-1)^n$

c) $t_n - t_{n-1} - 6t_{n-2} = 0$, $t_0 = 1$, $t_1 = 1$

Solution:

$r^3 - r - 6 = 0$

$(r\text{-}3)(r\text{+}2) = 0$, $\therefore r = 3, -2$

$\therefore t_n = c_1(3)^n + c_2(-2)^n$

Substituting the initial conditions one gets

$t_0 = c_1 + c_2 \therefore c_1 + c_2 = 1$

$t_1 = 3c_1 - 2c_2 \therefore\ 3c_1 - 2c_2 = 1$

Solving one get

$3c_1 + 3c_2 = 1$

$3c_1 - 2c_2 = 1$

$$5c_2 = 1 \Rightarrow c_2 = \frac{1}{5}$$

$$\therefore c_1 = \frac{4}{5}$$

Substituting, one get

$$t(n) = \left(\frac{4}{5}\right)3^n + \left(\frac{1}{5}\right)(-2)^n$$

4.7 Solve the following higher order recurrence equations.

a) $t_n - 3t_{n-1} + t_{n-2} - 3t_{n-3} = 0$

$t_0 = 0$; $t_1 = 1$; $t_2 = 1$

***Solution*:**

$r^3 - 3r^2 - r - 3 = 0$

$r = 3, 1, -1$

$\therefore t_n = c_1 3^n + c_2 1^n + c_3(-1)^n$

Substituting one get

$t_0 = c_1 + c_2 + c_3$

$t_1 = 3c_1 + c_2 + c_3$

$t_2 = 9c_1 + c_2 + 2c_3$

$$\therefore \quad c_1 = \frac{-1}{2}, \quad c_2 = \frac{1}{4}, \quad c_3 = \frac{3}{2}$$

Hence the general solution is $t_n = \left(\frac{-1}{4}\right)3^n + \frac{1}{4}(1)^n + \frac{3}{2}(1)^n$

b) $t_n - t_{n-1} + 2t_{n-2} - 6t_{n-3} = 0$

$$t_0 = 0 \; ; \; t_1 = 1 \; ; \; t_2 = 1$$

***Solution*:**

$$r^3 - r^2 + 2r + 6 = 0$$

4.8 Solve the following non-homogeneous equations

a) $t_n - 3t_{n-1} = 4^n(2n+1)$ for $n > 1$

$t_0 = 0 \; ; \; t_1 = 12$

Homogeneous part

$r - 3 = 0$

$\therefore r = 3$

Non-homogeneous part

$b^n(p(n))k \Longrightarrow 4^n(2n+1)$

$\Longrightarrow (r - b)^{k+1}$

$$= (r-4)^2$$

$\therefore r = 4, 4$

$\therefore \boxed{t_n = c_1 3^n + c_2 4^n + c_3 n 4^n}.$

b) $t_n - 5t_{n-1} + 7t_{n-2} - 3t_{n-3} = 1$, $n > 2$

$$t_0 = 1 \; ; \; t_1 = 2 \; ; \; t_2 = 3$$

***Solution*:**

Homogeneous part:

$r^3 - 5r^2 + 7r - 3 = 1$

$\therefore r = 1, 1, 3$

Non-homogeneous part:

$b^n(p(n))k \Longrightarrow 1^0$

$$(r-1)^{0+1} = (r-1) = \boxed{r = 1}\ .$$

The general solution is

$\therefore t_n = c_1 3^n + c_2 1^n + c_3 n \times 1^n + c_4 n^2 1^n$

Substituting the critical condition, one get

$c_1 + c_2 = 1$

$3c_1 + c_2 + c_3 + c_4 = 2$

The 3rd initial condition

$t_3 = 5$

Solving one gets

$c_1 = 2.125$

$c_2 = -1.125$

$c_3 = 1$

$c_4 = -4.25$

$\therefore t_n = 2.125 \times 3^n + (-1.125) \times 1^n + 1 \times n \times 1^n + (-4.25) n^2 \times 1^n$

$t_n = 2.125 \times 3^n - 1.125 + n - 4.25n^2$

4.9 Solve the following recurrence equations using generating functions.

a) $t_n - 4t_{n-1} = 0$; $t_0 = 1$

Solution:

Based on the problem (4.27), one can find

$$G(z) = \frac{1}{(1-4z)}$$

$\Rightarrow 1(1 + 4z + (4z)^2 + \cdots)$

$= 1 \times 4^n$

$= 4^n$

4.10 Write a recursive program for Fibonacci and solve it using generating functions.

Hint: $F_n = \frac{\phi^n - \psi^n}{\sqrt{5}} = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right].$

4.11 Convert the following sequence to generating functions.

a) 4^n

The sequence is $\{4, 16, 64, \cdots\}$

Based on problem 4.25, one can find

$$G(z) = \frac{1}{(1-4z)}$$

b) 2^n

Based on the preview problem, one can find that

$$G(z) = \frac{1}{(1-2z)}$$

4.12 Convert the following functions to partial functions

a) $\frac{x}{x^2+6x+9}$

b) $\frac{x-1}{x^2+10x+25}$

[Hint] For both these problems, there are double roots and hence partial fraction is difficult.

4.13 Verify the following functions are smooth or not

a) $n \log n$

Smooth function

b) x

Smooth function

4.14 Solve the following recurrence equations using domain and range transformation

a) T(n)= T($\sqrt{n}$) + n

b) T(n) = T^3($\sqrt{n}$) + n

4.15 For the principal amount \$100, if the compound interest is given by a bank is 3%. What would be recurrence equation ? Find the solution of the recurrence equation.

Solution:

Based on problem 4.11, one can find that the recurrence equation is $t_n = 1.03t_{n-1}$ and its solution is

$$\boxed{t_n = (1.03)t_0}\ .$$

4.16 Solve the following recurrence equation

$$t_n - 5t_{n-1} + 15t_{n-2} = 0$$

with initial conditions $t_0 = 0$, $t_1 = 1$ and $t_2 = 2$.

Solution:

$r^2 - 5r + 15 = 0$

$$= \frac{+5 \pm \sqrt{25-60}}{2}$$

$$= \frac{+5 \pm \sqrt{-35}}{2}$$

The roots are complex

And the general solution is

$$t_n = c_1\left(\frac{5+5.9i}{2}\right)^n + c_2\left(\frac{5-5.9i}{2}\right)^n$$

Solving this, we get

$c_1 + c_2 = 0$

$$c_1\left(\frac{5+5.9i}{2}\right) + c_2\left(\frac{5-5.9i}{2}\right) = 1$$

Multiply equation (1) by $\left(\frac{5+5.9i}{2}\right)$

$$c_1\left(\frac{5+5.9i}{2}\right) + c_2\left(\frac{5+5.9i}{2}\right) = 0$$

$$c_1\left(\frac{5+5.9i}{2}\right) + c_2\left(\frac{5-5.9i}{2}\right) = 1$$

Subtract (2) from (1), we get

$$c_2\left(\frac{5+5.9i}{2}\right)-c_2\left(\frac{5-5.9i}{2}\right)=1$$

$$\frac{5-5+5.9i+5.9i}{2}c_2=1$$

$$11.8ic_2=2$$

$$c_2=\frac{2}{11.8i}$$

$$\therefore c_1=\frac{-2}{11.8i}$$

$$\therefore t_n=\left(\frac{-2}{11.8i}\right)\left(\frac{5+5.9i}{2}\right)^n+\left(\frac{2}{11.8i}\right)\left(\frac{5-5.9i}{2}\right)^n$$

4.17 Use the simplified and generalized master theorem and solve the following :

a) $T(n)=T\left(\frac{n}{2}\right)+1$ Compare this with $T(n)=T\left(\frac{n}{2}\right)+c\times n^k$

Here $a=1$, $b=2$, $c=1$ and $k=0$

As $a=b^k$,

$$T(n)=\theta\left(n^k\log n\right)$$
$$=\theta(\log n).$$

b) $T(n)=8T\left(\frac{n}{2}\right)+10n^2$, $T(0)=0$

Using little master theorem

$a=8$, $b=2$, $k=2$

As $a>b^d$,

$$\theta\left(n^{\log_2 8}\right)=\theta\left(n^3\right)$$

Using master theorem

$$T(n) = 8T\left(\frac{n}{2}\right) + 10n^2 \quad , \quad T(0) = 0$$

as $a < b^k$

$$\therefore T(n) = \theta\left(n^{\log_2 8}\right) = \theta\left(n^3\right)$$

Note : one can note

$$T(n) = \begin{cases} \theta\left(n^k\right) & if \;\; a < b^k \\ \theta\left(n^k \log n\right) & if \;\; a = b^k \\ \theta\left(n^{\log_b a}\right) & if \;\; a > b^k \end{cases}$$

c) $T(n) = 3T\left(\frac{n}{2}\right) + n \quad , \quad T(0) = 0$

Using little master theorem

a = 8, b= 2 and k = 2

As $a>b^k$,

$$\therefore \theta\left(n^{\log_2 3}\right)$$

Using master theorem

$$T(n) = \theta\left(n^{\log_2 a}\right) = \theta\left(n^{\log_2 3}\right)$$

d) $T(n) = T\left(\frac{n}{2}\right) + (n-1)$

a = 1, b= 2, k = 1

$\therefore a<b^k$,

$$\therefore T(n) = \theta(n)$$

e) $T(n) = 2T\left(\frac{n}{2}\right) + n^2$

$a = 2$, $b = 2$, $k = 2$

$a < b^k$,

$\therefore \quad \theta(n^2)$.

Using master theorem

$$f(n) = \theta\left(n^{\log_2 2} \times n\right)$$
$$= \theta(n \times n)$$
$$= \theta\left(n^2\right)$$

$$T(n) = \theta(f(n)) = \theta\left(n^2\right)$$

f) $T(n) = T\left(\frac{n}{2}\right) + 1 \quad , \quad T(0) = 0$

***Solution*:**

$a = 7$, $b = 2$, $k = 0$

$a > b^k$,

$$\therefore = \theta\left(n^{\log_2 7}\right)$$

Using master theorem,

$$f(n) = n^{\log_2 \frac{7}{nt}}$$
$$= \theta\left(n^{2.8-2.8}\right) = \theta\left(n^0\right)$$

$$\therefore T(n) = \theta\left(n^{\log_2 7}\right)$$

g) $T(n) = 4T\left(\frac{n}{3}\right) + 1$

$a = 4$, $b = 3$, $k = 0$

$a > b^k$,

$$\therefore \theta\left(n^{\log_b a}\right)$$

$= \theta\left(n^{\log_3 4}\right)$

h) $T(n) = T\left(\frac{n}{2}\right) + n$

$a = 4$, $b = 3$, $k = 0$

$a < b^k$,

$\therefore \theta\left(n^k\right) = \theta(n)$

i) $T(n) = 7T\left(\frac{n}{2}\right) + 18\left(\frac{n}{2}\right)^2$

***Solution*:**

This is Stassen's multiplication algorithm

$7 > 2^2$

$\therefore \theta\left(n^{\log_2 7}\right)$

j) $T(n) = 3T\left(\frac{n}{3}\right) + n$

Solution:

Using little master theorem

$a = 4$, $b = 3$, $k = 0$

$a = b$,

$\therefore T(n) = \theta(n \log n)$

Using master theorem

$f(n) = \theta\left(n^{\log_3 3}\right)$

$= \theta(\log n)$

k) $T(n) = 2T\left(\frac{n}{2}\right) + (n-1)$

Solution:

$a = 2 = b^k$

$\therefore \theta\left(n^1 \log n\right)$

$= \theta\left(n \log n\right)$

(This is merge sort and closest pair algorithm)

l) $T(n) = 3T\left(\frac{n}{2}\right) + n$

$a = 3$, $b = 2$, $k = 1$

$3 > 2^1$

$\therefore T(n) = \theta\left(n^{\log_2 3}\right)$

m) $T(n) = 4T\left(\frac{n}{9}\right) + 5\sqrt{n}$

$a = 4$, $b = 9$, $k = ½$

$a > 9^{½} = 3$

$\therefore T(n) = \theta\left(\sqrt{n} \log n\right)$

n) $T(n) = 4T\left(\frac{n}{2}\right) + n \log n$

$a = 4$, $b = 2$, $k = 1$

$a > b^k$

$\therefore T(n) = \theta\left(n \log n\right)$

o) $T(n) = 3T\left(\frac{n}{4}\right) + n$

$a = 3$, $b = 4$, $k = 1$

$a < b^k$

$$\therefore T(n) = \theta\left(n^k\right) = \theta(n)$$

4.18 Solve the recurrence equation

$t_n - 3t_{n-1} = 2^n$

Solution:

Homogeneous part:

$r - 3 = 0$

$\therefore r = 3$

Non-homogeneous part

$b^n(p(n))^k = (r-2)^1$

$= r = 2$

$\therefore t(n) = c_1 3^n + c_2 2^n$

Assuming $t_0 = 0$, $t_1 = 1$, we get

$c_1 + c_2 = 0$

$3c_1 + 2c_2 = 0$

On solving we get

$3c_1 + 3c_2 = 0$

$3c_1 + 2c_2 = 0$

$\therefore c_2 = -1$ and $c_1 = 1$

$\therefore t(n) = 3^n - 2^n$.

4.19 Solve the recurrence equation

$$T(n) = 7T\left(\left\lfloor \frac{n}{4} \right\rfloor\right) \quad \text{where } T(1) = 1$$

Solution:

This is problem 4.34.

So the answer is

$$
\begin{aligned}
T(n) &= 7^{\log n} \\
&= n^{\log 7} \\
&= n^{3.81}
\end{aligned}
$$