# Chapter 17

17.1 Let there be three items A, B and C. The raw materials required for these products are $r_1$ and $r_2$. The daily allotment of raw materials is 90 and 80 respectively. How the raw materials should be allotted so that the profit is maximized.

| Product | Raw material | | Profit per unit |
|---|---|---|---|
| | $r_1$ | $r_2$ | |
| A | 2 | 2 | 140 |
| B | 3 | 5 | 120 |
| C | 5 | 5 | 250 |
| Daily Allotment | 10 | 14 | |

***Solution:***

Based on example 17.1, one can formulate LPP as

$3x_1 + 3x_3 + 5x_3 \leq 10$

$2x_1 + 5x_2 + 5x_3 \leq 14$

$x_1 , x_2 , x_3 \geq 0$

and the objective function is maximize

$Z = 140x_1 + 120x_2 + 250x_3$

17.2 There are three materials N1, N2 and N3. The need of a patient is at least 14 and 18. How to mix the following nutrients for the given cost consideration?

| Product | Raw material | | | Profit per unit |
|---|---|---|---|---|
| | $N_1$ | $N_2$ | $N_3$ | |
| $F_1$ | 6 | 2 | 4 | 9 |
| $F_2$ | 2 | 4 | 6 | 15 |
| $F_3$ | 5 | 3 | 7 | 23 |
| *Need* | 14 | 18 | 24 | – |

$6x_1 + 2x_3 + 5x_3 \leq 14$

$2x_1 + 4x_2 + 3x_3 \leq 18$

$4x_1 + 6x_2 + 7x_3 \leq 24$

$x_1, x_2, x_3 \geq 0$

and the objective condition is

$Z = 9x_1 + 15x_2 + 23x_3$

17.3 Formulate an LPP for finding the shortest path in a given graph.

***Solution:*** The input graph G is taken with source s and goal t. The edge weight represents the path from s to v.

The LPP formulation would be as follows: Let Xi be the distance between source and vertex i, then the LPP would be like

Minimize Xt

Subject to the conditions Xi.

17.4 Formulate an LPP for finding the minimum cost spanning tree in a graph.

***Solution***: The objective is to minimize the cost.

minimize $\sum_{i,j \in E} c_{ij} x_{ij}$ , Here c is the cost linking two vertices liked by edge (i,j). One constraint is, there are N-1 edges in T. The second constraint is any subset k involves only k-1 edges so that there is no cycle.

This is the structure of LPP problem.

17.5 Solve the following linear programming using the graphical method.

Maximize $Z = 2x_1 + 4x_3$

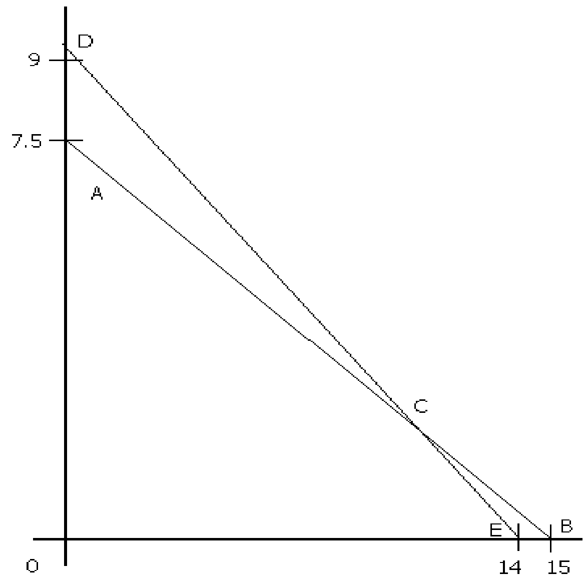
Subjected to

$x_1 + 2x_2 \leq 15$

$2x_1 + 3x_2 \leq 28$

$x_1, x_2 \geq 0$

***Solution*:**




| Point | X | Y | Value of Z |
|---|---|---|---|
| **0** | 0 | 0 | 0 |
| **A** | 0 | 7.5 | 30 |
| **B** | 15 | 0 | 30 |
| **C** | 11 | 2 | 30 |
| **D** | 0 | 9.33 | 37.33 |
| **E** | 14 | 0 | 23 |

17.6 Formulate a LPP model for maximum bipartite matching problem.

***Solution:*** Here, the vertices V is divided into two sets with edge connecting between vertices of two sets. The LPP formulation would be as follows:

One variable for every edge. For example, if vertex 1 is linked with A, then a variable is created as A1. One equality condition is created per vertex. The goal is to find the maximum cardinality matching.

So the problem shape is Maximize {Variables connecting Edges}

Subjected to All edges connected with vertex added with equality one.

17.7 Solve the following LPP using graphical method.

Maximize $Z = 24x_1 + 14x_2$

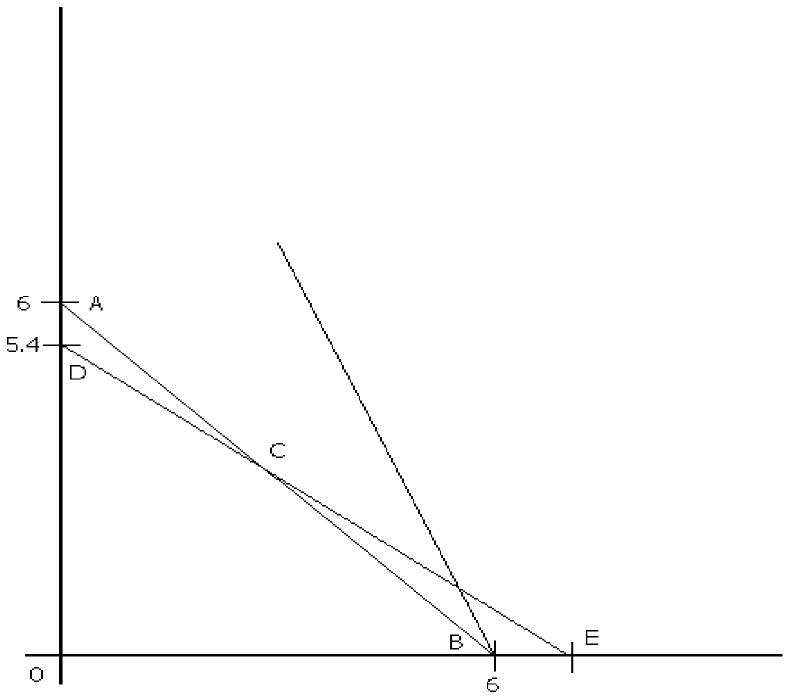
Subjected to

$8x_1 + 8x_2 \leq 48$

$9x_1 + 12x_2 \leq 64$

$x_1 , x_2 \geq 0.$

***Solution:***




| | X | Y | Value of Z |
|---|---|---|---|
| **0** | 0 | 0 | 0 |
| **A** | 0 | 6 | 8.40 |
| **B** | 6 | 0 | 14.40 |
| **C** | 2.66 | 3.33 | 110.66 |
| **D** | 0 | 5.33 | 74.66 |
| **E** | 7.11 | 0 | 170.66 |

∴ Answer is $x_1 = 7.1$ $x_2 = 0$

17.8 Solve the following LPP using graphical method.

Minimize $Z = 12x_1 + 8x_2$

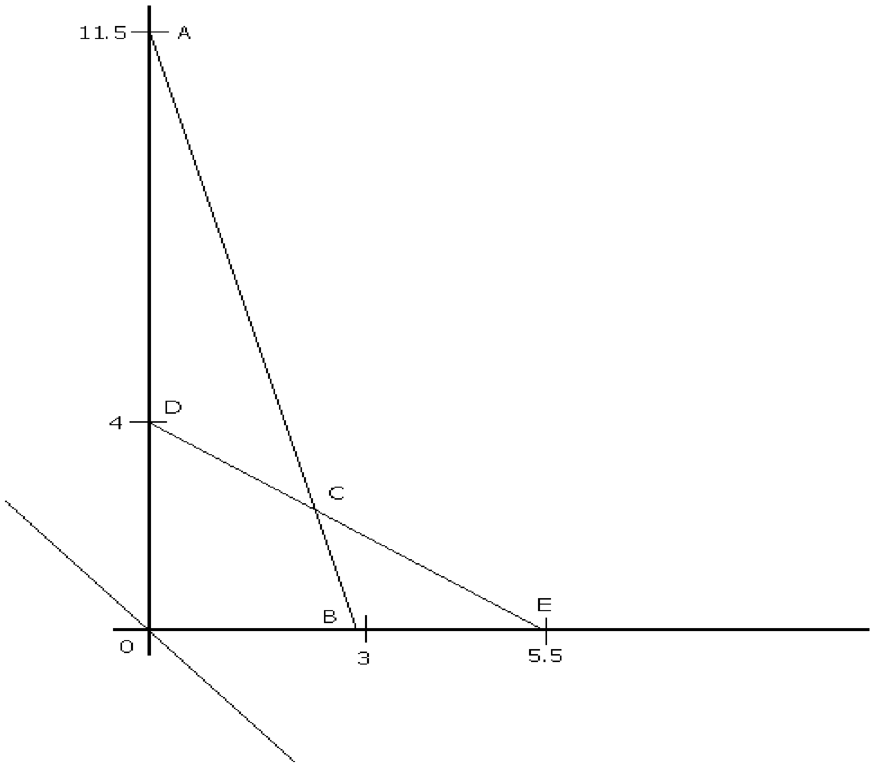
Subjected to

$8x_1 + 2x_2 \leq 23$

$9x_1 + 12x_2 \leq 48$

$x_1 , x_2 \geq 0.$

*Solution*:




| Point | X | Y | Value of Z |
|---|---|---|---|
| **0** | 0 | 0 | 0 |
| **A** | 0 | 11.5 | 92.00 |
| **B** | 2.87 | 0 | 34.50 |
| **C** | 2.31 | 2.27 | 45.85 |
| **D** | 0 | 4 | 32.00 |
| **E** | 5.33 | 0 | 64.00 |

∴ Answer is $x = 0$, $y = 4$.

17.9 Solve the following LPP using the simplex method.

Minimize $Z = 4x_1 + 3x_2 + 8x_3$

Subjected to $x_1 + 6x_2 + 12x_3 \leq 25$

$2x_1 + 8x_2 + 16x_3 \leq 42$

$x_1 , x_2 , x_3 \geq 0$

***Solution:***

$4x_1 + 3x_2 + 8x_3 + 0x_4 + 0x_5$

Subjected to

$x_1 + 6x_2 + 12x_3 + x_4 = 25$

$2x_1 + 8x_2 + 16x_3 + x_5 = 42$

$x_1 , x_2 , x_3 , x_4 , x_5 \geq 0$

| **Table** | | | **4** | **3** | **8** | **0** | **0** |
|---|---|---|---|---|---|---|---|
| **Base** | $C_b$ | $p_0$ | $p_1$ | $p_2$ | $p_3$ | $p_4$ | $p_5$ |
| $p_4$ | 0 | 25 | 1 | 6 | $\boxed{12}$ | 1 | 0 |
| $P_5$ | 0 | 42 | 3 | 8 | 16 | 0 | 1 |
| $Z$ | 0 | -4 | -3 | -8 | 0 | 0 | 0 |

The leaving variable is $p_4$ and entering variable is $p_3$

| **Table** | | | **4** | **3** | **8** | **0** | **0** |
|---|---|---|---|---|---|---|---|
| **Base** | $C_b$ | $p_0$ | $p_1$ | $p_2$ | $p_3$ | $p_4$ | $p_5$ |
| $P_3$ | 8 | $\frac{25}{12}$ | $\frac{1}{12}$ | $\frac{1}{2}$ | 1 | $\frac{1}{12}$ | 0 |
| $P_5$ | 0 | $\frac{26}{3}$ | $\boxed{\frac{2}{3}}$ | 0 | 0 | $\frac{4}{3}$ | 1 |
| $Z$ | 0 | $\frac{-50}{3}$ | $\frac{-10}{3}$ | 1 | 0 | $\frac{2}{3}$ | 0 |

The leaving variable is $p_5$ and entering variable is $p_1$

| **Table 3** | | | **4** | **3** | **8** | **0** | **0** |
|---|---|---|---|---|---|---|---|
| **Base** | $C_b$ | $p_0$ | $p_1$ | $p_2$ | $p_3$ | $p_4$ | $p_5$ |
| $P_3$ | 8 | 1 | 0 | $\frac{1}{2}$ | 1 | $\boxed{\frac{1}{4}}$ | $\frac{-1}{8}$ |
| $P_1$ | 4 | 13 | 1 | 0 | 0 | -2 | $\frac{3}{2}$ |
| $Z$ | 0 | 60 | 0 | 1 | 0 | -6 | 5 |

The leaving variable is $p_3$ and entering variable is $p_4$

| **Table 4** | | | **4** | **3** | **8** | **0** | **0** |
|---|---|---|---|---|---|---|---|
| **Base** | $C_b$ | $p_0$ | $p_1$ | $p_2$ | $p_3$ | $p_4$ | $p_5$ |
| $P_4$ | 0 | 4 | 0 | 2 | 4 | 1 | $\frac{-1}{2}$ |
| $P_1$ | 4 | 21 | 1 | 4 | 8 | 0 | $\frac{1}{2}$ |
| $Z$ | 0 | 84 | 0 | 13 | 24 | 0 | 2 |

∴ The Optimal solution is $z = 84$

$x_1 = 21$

$x_2 = 0$

$x_3 = 0$

17.10 Solve the following LPP using simplex method

Maximize $Z = 12x_1 + 24x_2 + 23x_3 + x_4$

Subjected To $2x_1 + 4x_2 + 5x_3 + 6x_4 \leq 43$

$3x_1 + 4x_2 + 5x_3 + 5x_4 \leq 64$

$x_1, x_2, x_3, x_4 \geq 0$

***Solution:***

$$12x_1 + 24x_2 + 23x_3 + x_4 + 0x_5 + x_6$$

$$2x_1 + 4x_2 + 5x_3 + 6x_4 + x_5 = 43$$

$$3x_1 + 4x_2 + 5x_3 + 5x_4 + x_6 = 64$$

$x_1, x_2, x_3, x_4 \geq 0$

| **Table 1** | | | **12** | **24** | **23** | **1** | **0** | **0** |
|---|---|---|---|---|---|---|---|---|
| **Base** | $C_b$ | $p_0$ | $p_1$ | $p_2$ | $p_3$ | $p_4$ | $p_5$ | $P_6$ |
| $P_5$ | 0 | 43 | 2 | 4 | 5 | 6 | 1 | 0 |
| $P_6$ | 0 | 64 | 3 | 4 | 5 | 5 | 0 | 1 |
| $Z$ | 0 | 0 | -12 | -21 | -23 | -1 | 0 | 0 |

The leading is $p_5$and entering variable $p_2$**.**

| **Table 2** | | | **12** | **24** | **23** | **1** | **0** | **0** |
|---|---|---|---|---|---|---|---|---|
| **Base** | $C_b$ | $p_0$ | $p_1$ | $p_2$ | $p_3$ | $p_4$ | $p_5$ | $P_6$ |
| $P_2$ | 24 | 10.25 | 0.5 | 1 | 1.25 | 1.5 | 0.25 | 0 |
| $P_6$ | 0 | 21 | 1 | 0 | 0 | -1 | -1 | 1 |
| $Z$ | 0 | 258 | 0 | 0 | 7 | 35 | 6 | 0 |

There is infinitely many values of $x_1$ , $x_2$ , $x_3$ , $x_4$ for the optimal value of Z = 258, which are contained in the region of space $12x_1 + 24x_2 + 23x_3 + x_4 = 258$ that satisfies the constraints.

Answer is $x_1 = 0$

$x_2 = 10.75$

$x_3 = 0$

$x_4 = 0$

17.11 Solve the following LPP using the simplex method

$Z = 8x_1 + 9x_2 + 2x_3$

Subjected To

$3x_1 + 4x_2 + 5x_3 \leq 16$

$2x_1 + 3x_2 + x_3 \leq 24$

$x_1, x_2, x_3 \geq 0$

***Solution:***

Maximize $8x_1 + 9x_2 + 2x_3 + x_4 + x_5$

Subjected to

$3x_1 + 4x_2 + 5x_3 + x_4 = 16$

$2x_1 + 3x_2 + x_3 + x_5 = 24$

$x_1, x_2, x_3, x_4, x_5 \geq 0$

| **Table 1** | | | **8** | **9** | **2** | **0** | **0** |
|---|---|---|---|---|---|---|---|
| **Base** | $C_b$ | $p_0$ | $p_1$ | $p_2$ | $p_3$ | $p_4$ | $p_5$ |
| $P_4$ | 0 | 16 | 3 | $\boxed{4}$ | 5 | 1 | 0 |
| $P_5$ | 0 | 24 | 2 | 3 | 1 | 0 | 1 |
| $Z$ | | 0 | -5 | -9 | -2 | 0 | 0 |

The leaving variable is $p_4$and entering variable is $p_2$.

| **Table 2** | | | **8** | **9** | **2** | **0** | **0** |
|---|---|---|---|---|---|---|---|
| **Base** | $C_b$ | $p_0$ | $p_1$ | $p_2$ | $p_3$ | $p_4$ | $p_5$ |
| $P_2$ | 9 | 4 | $3/4$ | 1 | $5/4$ | $1/4$ | 0 |
| $P_5$ | 0 | 12 | $-1/4$ | 0 | $-11/4$ | $-3/4$ | 1 |
| $Z$ | | 36 | $-5/4$ | 0 | $37/4$ | $9/4$ | 0 |

The leaving variable is $p_2$ and entering variable is $p_1$.

| **Table 3** | | | **8** | **9** | **2** | **0** | **0** |
|---|---|---|---|---|---|---|---|
| **Base** | $C_b$ | $p_0$ | $p_1$ | $p_2$ | $p_3$ | $p_4$ | $p_5$ |
| $P_4$ | 8 | $16/3$ | 1 | $4/3$ | $5/3$ | $1/3$ | 0 |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| ***P*$_5$** | 0 | $^{40}/_3$ | 0 | $^{1}/_3$ | $^{-7}/_3$ | $^{-2}/_3$ | 1 |
| ***Z*** | 0 | $^{128}/_3$ | 0 | $^{5}/_3$ | $^{34}/_3$ | $^{8}/_3$ | 0 |

∴ The optimal solution is

$Z = \dfrac{128}{3}$

$x_1 = \dfrac{16}{3}$

$x_2 = 0$

$x_3 = 0$

17.12 Solve the following LPP using the two–table simplex method.

Minimize $Z = 2x_1 + 4x_2 + 3x_3$

Subjected to

$x_1 + x_1 + x_3 \leq 18$

$x_1 + 3x_2 + 2x_3 \leq 28$

$x_1, x_2, x_3 \geq 0$

***Solution*:**

Maximize $-2x_1 - 4x_2 - 3x_3 + x_4 + x_5 + x_6$

Subjected to

$x_1 + x_1 + x_3 + x_4 = 18$

$x_1 + 3x_2 + 2x_3 + x_5 = 28$

$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$

Construct phase 1

| **Table 1** | | | **0** | **0** | **0** | **0** | **-1** | **-1** |
|---|---|---|---|---|---|---|---|---|
| **Base** | ***C*$_b$** | ***p*$_0$** | ***p*$_1$** | ***p*$_2$** | ***p*$_3$** | ***p*$_4$** | ***p*$_5$** | ***P*$_6$** |
| ***P*$_6$** | -1 | 18 | 1 | 1 | 1 | -1 | 0 | 1 |
| ***P*$_5$** | -1 | 28 | 1 | 3 | 2 | 0 | 1 | 0 |
| ***Z*** | 0 | -46 | -2 | -4 | -3 | 1 | 0 | 0 |

The leaving variable is $p_5$ and entering variable is $p_2$.

| **Table 2** | | | **0** | **0** | **0** | **0** | **-1** | **-1** |
|---|---|---|---|---|---|---|---|---|
| **Base** | $C_b$ | $p_0$ | $p_1$ | $p_2$ | $p_3$ | $p_4$ | $p_5$ | $P_6$ |
| $P_6$ | -1 | $26/3$ | $2/3$ | 0 | $1/3$ | -1 | $-1/3$ | 1 |
| $P_2$ | 0 | $25/3$ | $1/3$ | 1 | $2/3$ | 0 | $1/3$ | 0 |
| $Z$ | 0 | $-26/3$ | $-2/3$ | 0 | $-1/3$ | 1 | $4/3$ | 0 |

The leaving variable is $p_6$ and entering variable is $p_1$ .

| **Table 3** | | | **0** | **0** | **0** | **0** | **-1** | **-1** |
|---|---|---|---|---|---|---|---|---|
| **Base** | $C_b$ | $p_0$ | $p_1$ | $p_2$ | $p_3$ | $p_4$ | $p_5$ | $P_6$ |
| $P_1$ | 0 | 13 | 1 | 0 | $1/2$ | $-3/2$ | $-1/2$ | $3/2$ |
| $P_2$ | 0 | 5 | 0 | 1 | $1/2$ | $1/2$ | $1/2$ | $-1/2$ |
| $Z$ | | 0 | 0 | 0 | 0 | 0 | 1 | 1 |

There may be any solution, so one can continue with phase II to calculate it.

| **Table 1** | | | **-2** | **-4** | **-3** | **0** |
|---|---|---|---|---|---|---|
| **Base** | $C_b$ | $p_0$ | $p_1$ | $p_2$ | $p_3$ | $p_4$ |
| $P_1$ | -2 | 13 | 1 | 0 | $1/2$ | $-3/2$ |
| $P_2$ | -4 | 5 | 0 | 1 | $1/2$ | $1/2$ |
| $Z$ | | -46 | 0 | 0 | 0 | 1 |

There are many solutions. One solution is

$x_1 = 13$

$x_2 = 5$

$x_3 = 0$

17.13 Find the dual at the following LPP and verify it

Maximize $Z = 2x_1 + 4x_2$

Subjected to

$x_1 + 2x_2 \leq 15$

$2x_1 + 3x_2 \leq 28$

$x_1, x_2 \geq 0$

***Solution*:**

The dual problem can be given as

Minimize $Z' = 15y_1 + 28y_2$

Subjected to the condition

$y_1 + 2y_2 \leq 2$

$2y_1 + 3y_2 \leq 4$

$y_1, y_2 \geq 0.$

The verification part is, the dual of the dual is its primal. It is the proof.

17.14 Find the dual of the following LPP and verify it.

Maximize $Z = 12x_1 + 8x_2$

Subjected to

$8x_1 + 2x_1 \leq 23$

$9x_1 + 12x_2 \leq 48$

$x_1, x_2 \geq 0$

***Solution*:**

The dual problem can be written as

Minimize $Z = 23y_1 + 48y_2$

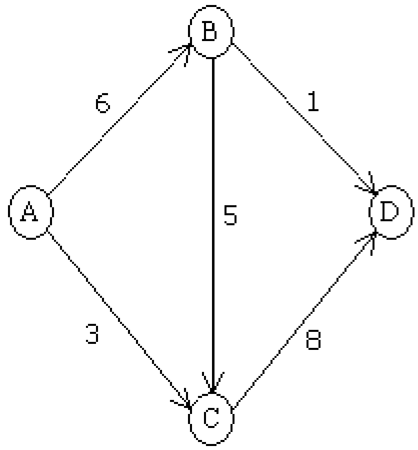
$8y_1 + 9y_2 \leq 12$

$2y_1 + 12y_2 \leq 8$

$y_1, y_2 \geq 0$

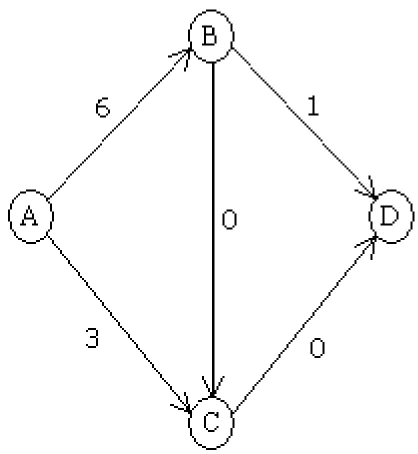
Hence $y_1, y_2$ is unrestricted with respect to sign.

17.15 Solve the following problem using Ford–Fulkerson algorithm.




***Solution*:**

The initial flow graph is given as follows.



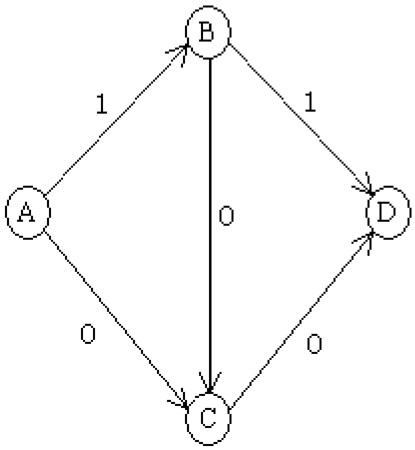

The commodity can be increased by 3.

Construct one augment path A⟶B⟶D.

| Edges | Total Capacity | Current Load | Excess Capacity |
|---|---|---|---|
| **A → B** | 6 | 0 | 6 |
| **B → D** | 0 | 1 | 1 |

Minimum { 1, 6 } = 1

∴ Increase the load by 1.

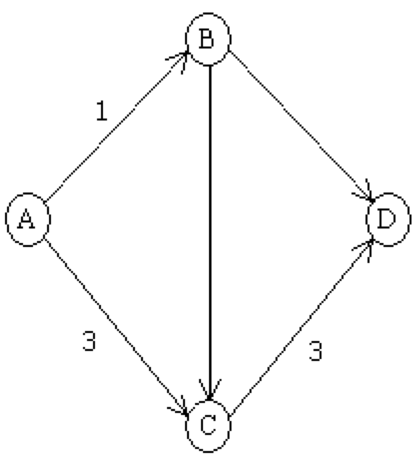



Next augment path is ACD

| **Edges** | **Total Capacity** | **Current Load** |
|---|---|---|
| **A → C** | 3 | 0 |
| **C → D** | 8 | 0 |

Minimum { 3, 8 } = 3

∴ The updated graph would be

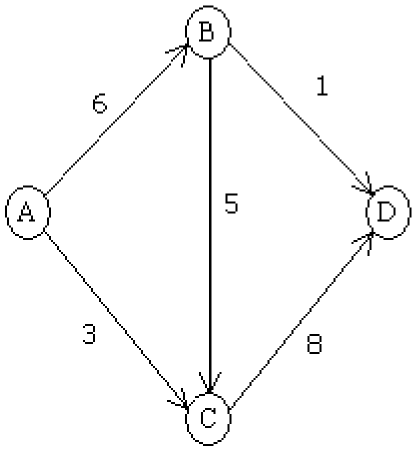



The next augment path is ABCD.

| **Edges** | **Total Capacity** | **Current Load** | **Excess Capacity** |
|---|---|---|---|
| **A → B** | 6 | 1 | 5 |
| **B →C** | 5 | 0 | 5 |

| | | | |
|---|---|---|---|
| **C→D** | 8 | 3 | 5 |

Minimum { 5, 5, 5 }

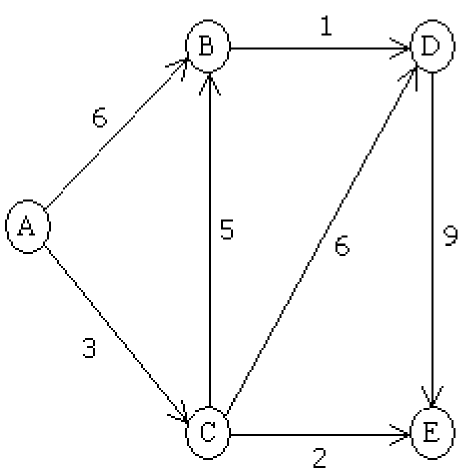



Incidentally the maximum flow is same on the original graph.

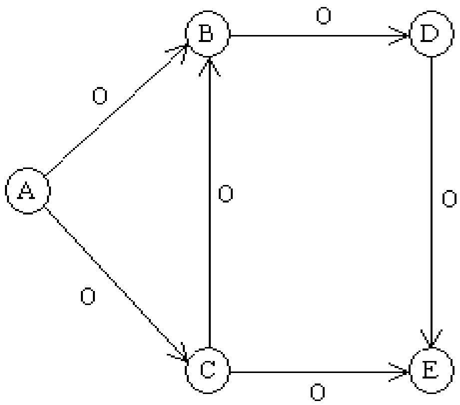
17.16 Solve the following problem.

***Solution*:**

The initial graph is




The initial augment path is



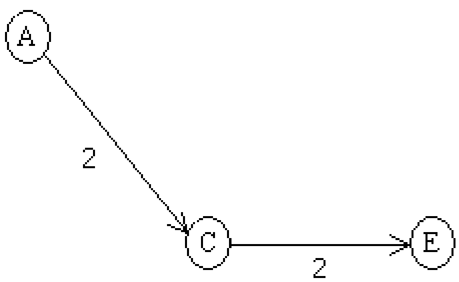

A C E

The minimum is completed as A——→C

| Edges | Total Capacity | Current Load |
|---|---|---|
| **A → C** | 3 | 0 |
| **C → D** | 2 | 0 |

Minimum { 2, 3 } = 2.

∴ Increase capacity by 2 to get this graph.

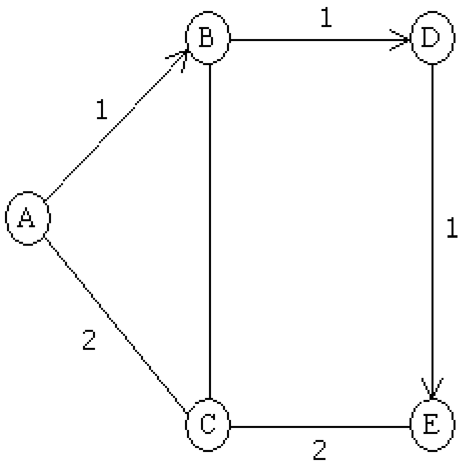



The next augment path is A B D E

| Edges | Total Capacity | Current Load | Excess Capacity |
|---|---|---|---|
| **A → B** | 6 | 0 | 6 |
| **B → D** | 1 | 0 | 1 |
| **B→E** | 9 | 0 | 9 |

Minimum { 6, 1, 9 } = 1.

∴ Increase by 1.

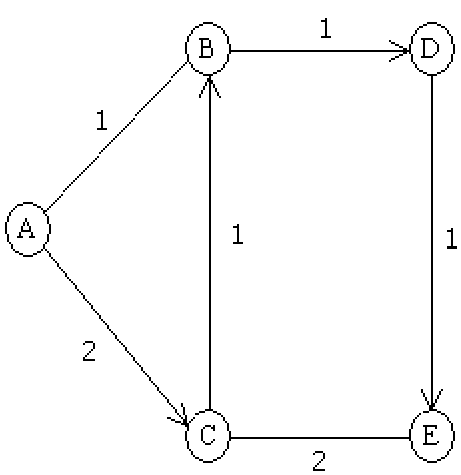



The next augment path is A C B D E

| Edges | Total Capacity | Current Load | Excess Capacity |
|---|---|---|---|
| **A →C** | 3 | 2 | 1 |
| **C→B** | 5 | 0 | 5 |
| **B→D** | 1 | 1 | 0 |
| **D→E** | 9 | 1 | 8 |

Minimum is {0}

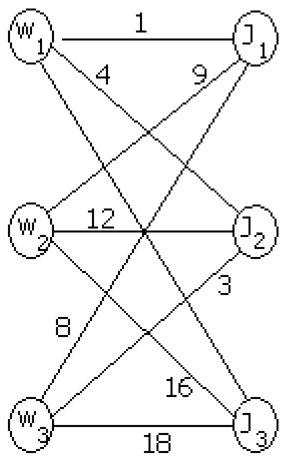
The final graph is




The minimum flow is 3.

17.17 Let there be 3 workers and three jobs, the cost matrix for which is as follows:

| Workers \ Jobs | 1 | 2 | 3 |
|---|---|---|---|
| **1** | 2 | 4 | 8 |
| **2** | 9 | 12 | 16 |
| **3** | 1 | 3 | 18 |




One can apply Hungarian method now.

17.18 Solve the stable marriage problem for 3 boys and 3 girls with the following pref. matrix.

| | | | |
|---|---|---|---|
| $B_0$ | $G_0$ | $G_2$ | $G_1$ |
| $B_1$ | $G_2$ | $G_0$ | $G_1$ |
| $B_2$ | $G_2$ | $G_1$ | $G_0$ |
| $G_0$ | $B_1$ | $B_0$ | $B_2$ |
| $G_1$ | $B_2$ | $B_1$ | $B_0$ |
| $G_2$ | $B_1$ | $B_0$ | $B_2$ |

Show the stable matching pairs.

***Solution*:**

Round I

| | |
|---|---|
| $B_0$ | prefers $G_0$ |
| $G_0$ | is unmatched |
| $B_0$ | becomes pair with $G_0$ |
| $B_1$ | prefers $G_2$ |
| $G_2$ | previously unmatched |
| $B_1$ | pair with $G_2$ |
| $B_2$ | prefers $G_2$ |
| $G_2$ | is already paired with $B_1$ |
| $G_2$ | rank $B_1$ as 0 |
| $G_2$ | rank $B_2$ as 2 |
| $B_2$ | moves on to $G_1$ |

RoundII

| | |
|---|---|
| $B_0$ | prefers $G_0$ |
| $G_0$ | already matched to $B_0$ |
| $B_1$ | prefers $G_2$ |
| $G_2$ | is already matched to $B_1$ |
| $B_2$ | prefers $G_1$ |
| $G_1$ | is previously unmatched |
| $B_2$ | becomes pair with $G_1$ |

∴ Result:

$B_0$ with $G_0$

$B_1$with $G_2$

$B_2$with $G_1$