Chapter 13

13.1 Compute the following binomial coefficients using the dynamic programming approach.

a) *B*[4, 2]

***Solution*:**

0^{th} row : *B*[0, 0]

1^{st} row : *B*[1, 0] = 1, *B*[1, 1] = 1

2^{nd} row : *B*[2, 0] = 1, *B*[2, 1] = 2, *B*[2, 2] = 2

So the Pascal triangle is formed as

1

1 1

1 2 1

The 3^{rd} row would be 1, 3, 3, 1 and

4^{th} row would be 1, 4, 6, 4, 1

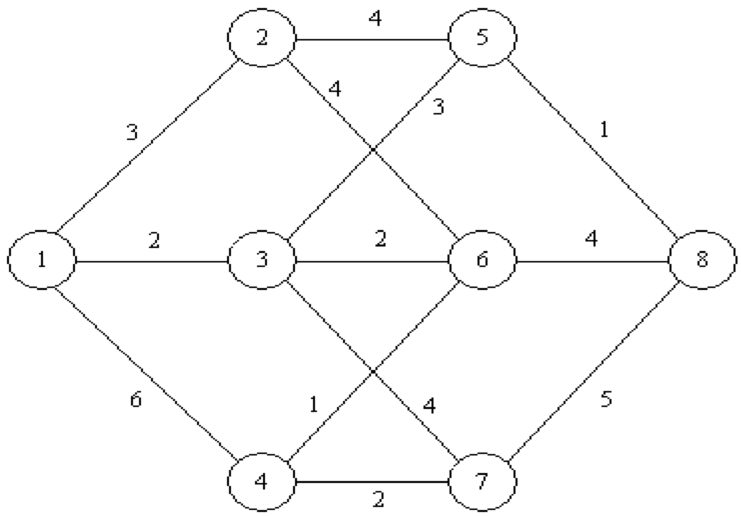
B[4, 2] = 6.

a) *B*[5, 3]

The 5^{th} row would be 1, 5, 10, 10, 5, 1

B[5, 3] = 10.

13.2 Solve the following multistage problem using both forward and backward reasoning .



***Solution*:**

Stage 3 – 4

Cost(3,5) = 1

Cost(3,6) = 4

Cost(3,7) = 5

Stage 2 – 3

$$Cost(2,2) = \min \begin{cases} 4 + Cost(3,5) = 4 + 1 = 5 \\ 3 + Cost(3,5) = 3 + 1 = 4 \end{cases}$$

$$Cost(2,3) = \min \begin{cases} 4 + Cost(3,6) = 4 + 4 = 8 \\ 2 + Cost(3,6) = 2 + 4 = 6 \\ 1 + Cost(3,6) = 1 + 4 = 5 \end{cases}$$

$$Cost(2,4) = \min \begin{cases} 4 + Cost(3,7) = 4 + 5 = 9 \\ 2 + Cost(3,7) = 2 + 5 = 7 \end{cases}$$

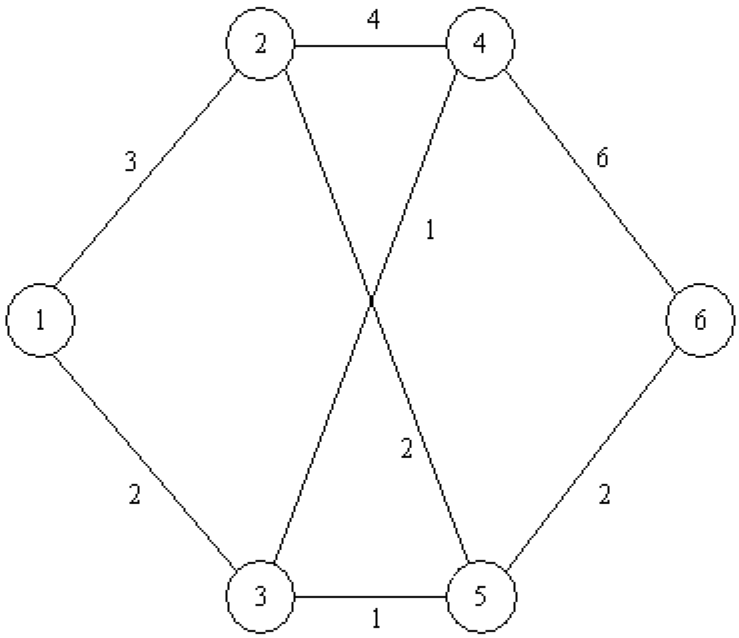
Stage 1 – 2

$$Cost(1,1) = \min \begin{cases} 3 + Cost(2,2) = 3 + 4 = 7 \\ 2 + Cost(2,3) = 2 + 5 = 7 \\ 6 + Cost(2,4) = 6 + 7 = 13 \end{cases}$$

$\therefore$ 1 – 2 – 5 – 8 is the shortest path.

(One more path is 1 – 3 – 6 – 8)

13.3 Solve the following multistage problem using both forward and backward reasoning.



Solution:

There are four stages

Stage 3 – 4

Cost (3,4) = 6

Cost (3,5) = 2

Stage 2 – 3

$$Cost(2,2) = \min\begin{cases} 4 + Cost(3,4) = 4 + 6 = 10 \\ 1 + Cost(3,4) = 1 + 6 = 7 \end{cases}$$

$$Cost(2,3) = \min\begin{cases} 1 + Cost(3,5) = 1 + 2 = 3 \\ 2 + Cost(3,5) = 2 + 2 = 4 \end{cases}$$

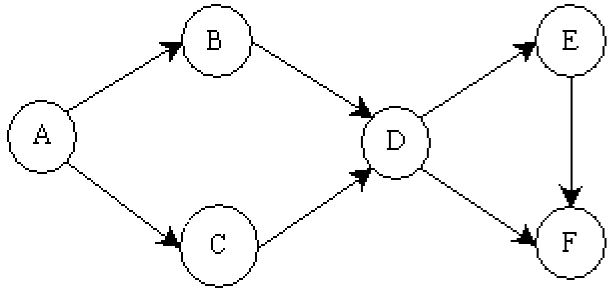
Stage 1 – 2

$$Cost(1,1) = \min\begin{cases} 3 + Cost(2,2) = 3 + 7 = 10 \\ 2 + Cost(2,3) = 2 + 3 = 5 \end{cases}$$

The path is 1 – 3 – 5 – 6 with cost of 5.

13.4 Use Warshall algorithm to create transitive closures of the following digraph. Show the intermediate results

a)



Solution:

The initial adjacency matrix is given as

	A	B	C	D	E	F
A	0	1	1	0	0	0
B	0	0	0	1	0	0
C	0	0	0	1	0	0
D	0	0	0	0	1	1
E	0	0	0	0	0	1
F	0	0	0	0	0	0

Add A as an active vertex. The resultant path matrix $P^{(1)}$ is same as the adjacency vertex. In $P^{(2)}$, the vertex B is made active. So path $A \to D$ is made possible. The path matrix would be come

$$\begin{array}{c c}
 & \begin{array}{cccccc} A & B & C & D & E & F \end{array} \\
\begin{array}{c} A \\ B \\ C \\ D \\ E \\ F \end{array} & \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}
\end{array}$$

In $P^{(3)}$, the vertex C is made active. The path matrix is same as the above.

In $P^{(4)}$, the vertex D is made active. As a result the vertex E becomes accessible to A and C. Similarly, F is become accessible to A and C. $\therefore$ The path matrix would become as

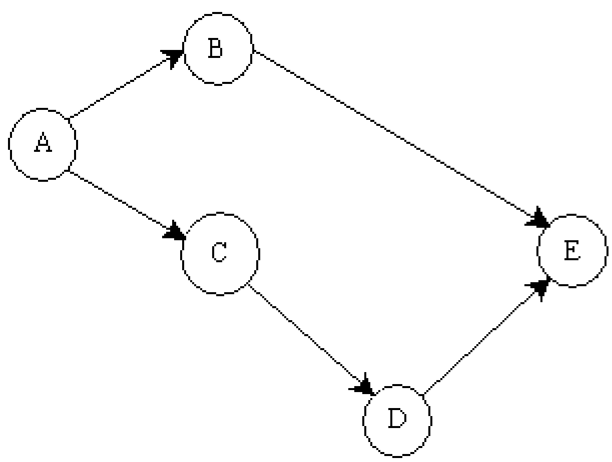
$$\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \\ F \end{array}\begin{array}{c} \begin{array}{cccccc} A & B & C & D & E & F \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

In P(5), the vertex E is made active. The path matrix is similar to above. In the last iteration P(6), all nodes are made active.

The final path matrix is given as

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

b)



***Solution*:**

The initial adjacency matrix is given as

$$\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array}\begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

In $P^{(1)}$, the node A is made active. So the path matrix is same as before. In $P^{(2)}$, the node B is made active. So there is a path between A and E. The path matrix would become as

$$\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array}\begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

In $P^{(3)}$, node C is made active. The path matrix is same as above. In $P^{(4)}$, the node D becomes active. In $P^{(5)}$, the node E becomes active. ∴ The path between B to D and C are possible.

∴ The path matrix is

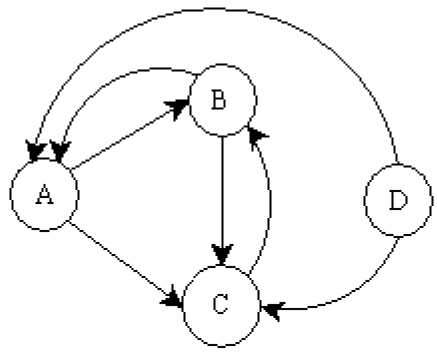
$$\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array}\begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

It can be observed that node ‘C’ is isolated node.

13.5 The following is the adjacency matrix of a digraph. Draw the graph and construct the transitive closure using the alternative version of Warshall’s algorithm.

$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

***Solution*:**



To compute $P^{(1)}$, check column 1 of the adjacency matrix. There is '1' in row 2 and row 4. So replace row 2 and row 4 as follows.

$$\begin{array}{r} (1 \quad 0 \quad 1 \quad 0) \\ \vee(0 \quad 1 \quad 1 \quad 0) \\ \hline 1 \quad 1 \quad 1 \quad 0 \\ \hline \end{array}$$

$$\begin{array}{r} (1 \quad 0 \quad 1 \quad 0) \\ \cup(0 \quad 1 \quad 1 \quad 0) \\ \hline 1 \quad 1 \quad 1 \quad 0 \\ \hline \end{array}$$

$\therefore$ The matrix would become as

$$\begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

Check the second column. There is '1' is all the rows. So replace all rows with ∨ of row 2. This yield

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

Check the 3rd column. There is '1' is all the rows. So replace all rows with $\vee$ of row 2 to get

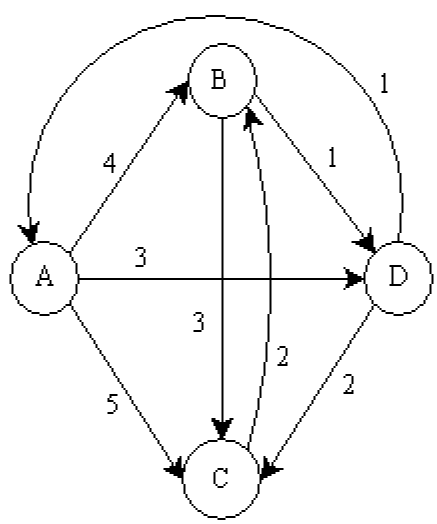
$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

Check the 4th column. There is no '1'. So Stop. So it can be observed that 'D' is an isolated node.

13.6 Use the Floyd-Warshall's algorithm and find all pairs shortest path for the adjacency weighted matrix.

$$\begin{array}{c c} & \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{array}{c} A \\ B \\ C \\ D \end{array} & \begin{pmatrix} 0 & 14 & \infty & 0 \\ \infty & 0 & 2 & 1 \\ 5 & 3 & 0 & \infty \\ 1 & \infty & 2 & 0 \end{pmatrix} \end{array}$$

Solution :



$D^{(0)}$ is same as the adjacency matrix. In the first iteration A is made active. So path D to B is possible. The resultant matrix would be

$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{pmatrix} 0 & 4 & \infty & 3 \\ \infty & 0 & 2 & 1 \\ 5 & 3 & 0 & \boxed{8} \\ 1 & \boxed{5} & 2 & 0 \end{pmatrix} \end{array}$$

In D(2), the node B is also made active. The resultant matrix would be

$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{pmatrix} 0 & 4 & \boxed{6} & 3 \\ \infty & 0 & 2 & 1 \\ 5 & 3 & 0 & 8 \\ 1 & 5 & 2 & 0 \end{pmatrix} \end{array}$$

In D(3), the matrix would become as

$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{pmatrix} 0 & 4 & 6 & 3 \\ \boxed{7} & 0 & 2 & 1 \\ 5 & 3 & 0 & 2 \\ 1 & 5 & 2 & 0 \end{pmatrix} \end{array}$$

In D(4), the node D becomes active. $\therefore$ The resultant matrix would become as

$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{pmatrix} 0 & 4 & \boxed{5} & 3 \\ 7 & 0 & 2 & 1 \\ 5 & 3 & 0 & 2 \\ 1 & 5 & 2 & 0 \end{pmatrix} \end{array}$$

This is the final cost matrix.

13.7 Solve the following chain matrix multiplication problem.

a)
$$\begin{array}{cccc} A & B & C & D \\ (2,3) & (3,4) & (4,5) & (5,6) \end{array}$$

Solution:

1	2	3	4

A	0	27	64	124
B		0	60	150
C			0	120
D				0

The associated table would be

	2	3	4
A	1	2	3
B		2	3
C			3
D			

∴ The minimum cost is 124 and the order is [A (B C) D].

b)

A	B	C	D	E
2×4	4×6	6×8	8×12	12×6

Solution :

	1	2	3	4	5
A	0	48	144	336	480
B		0	192	576	864
C			0	576	864
D				0	576
E					0

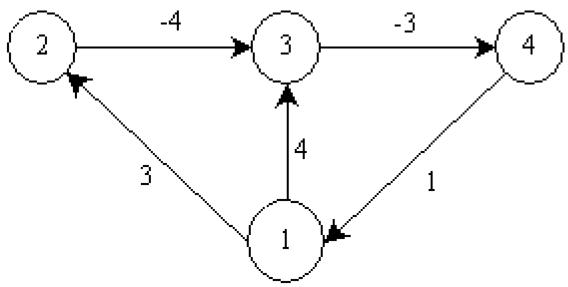
And the associated table is

	2	3	4	5
A	1	2	3	4
B		2	3	4
C			3	3

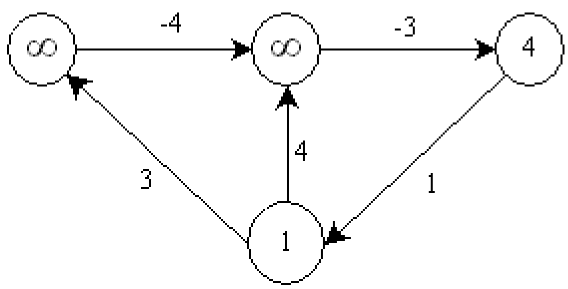
D				4
E				

∴ The minimum cost is 480 and order is {[A (B C) D] E}.

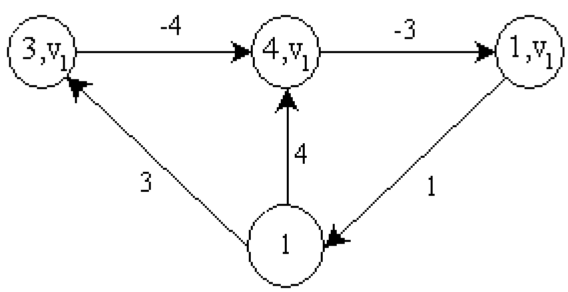
13.8 Use the Bellman-Ford algorithm and find the shortest path between nodes 1 and 3.

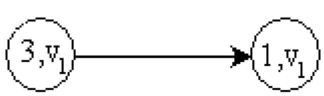


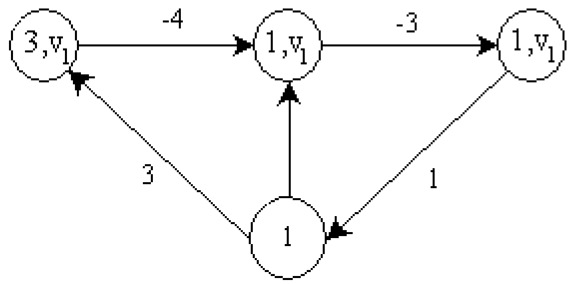
***Solution*:**



Relax the labels

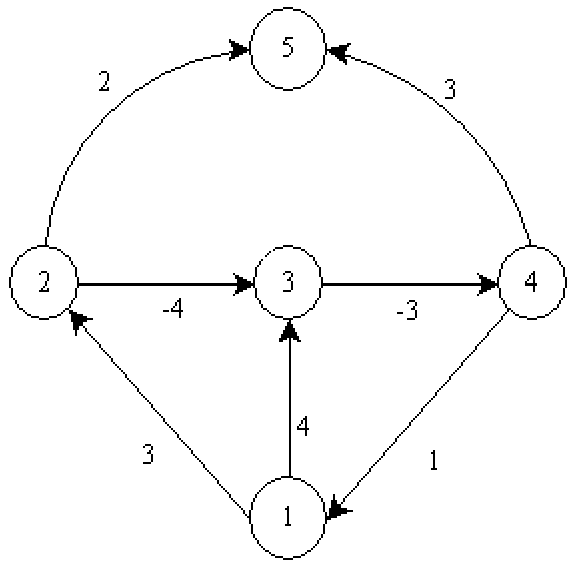


Relax the label as  to get

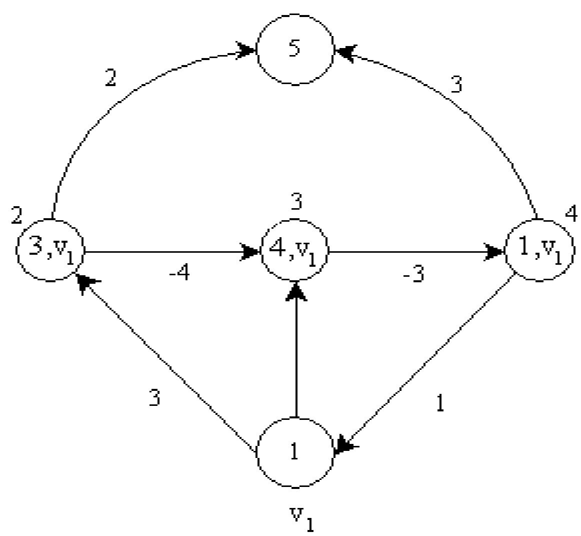


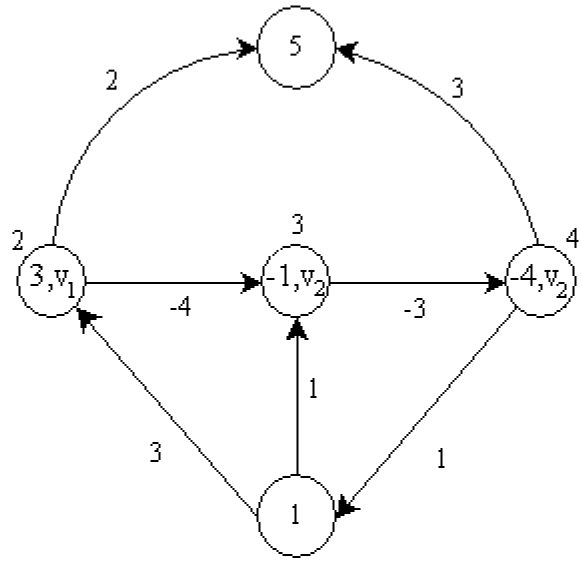
∴ The shortest path between (1,3) is 1 – 2 – 3.

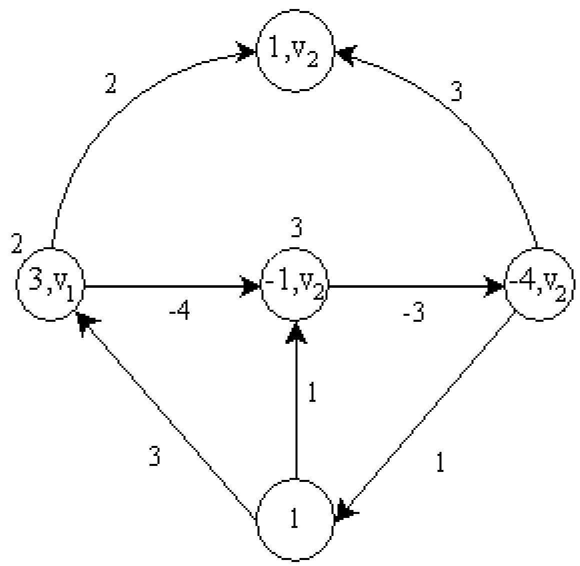
13.9 Use the Bellman-Ford algorithm and find shortest path between nodes 1 and 5.



***Solution*:**







The shortest path is 1, 2, 3, 4, 5 with path cost of 1.

13.10 Consider the following weighted adjacency matrix and draw graph first. Then, dynamic programming to the constructed graph and find the optimal travelling salesperson tour. Assume that the starting vertex is 1.

$$\begin{pmatrix} 0 & 3 & 5 & 10 \\ 2 & 0 & 4 & 8 \\ 4 & 2 & 0 & 8 \\ 5 & 6 & 3 & 0 \end{pmatrix}$$

$$Cost(1,\{2,3,4\}) = \min \begin{cases} d[1,2] + Cost(2,\{3,4\}) \\ d[1,3] + Cost(3,\{2,4\}) \\ d[1,4] + Cost(4,\{2,3\}) \end{cases}$$

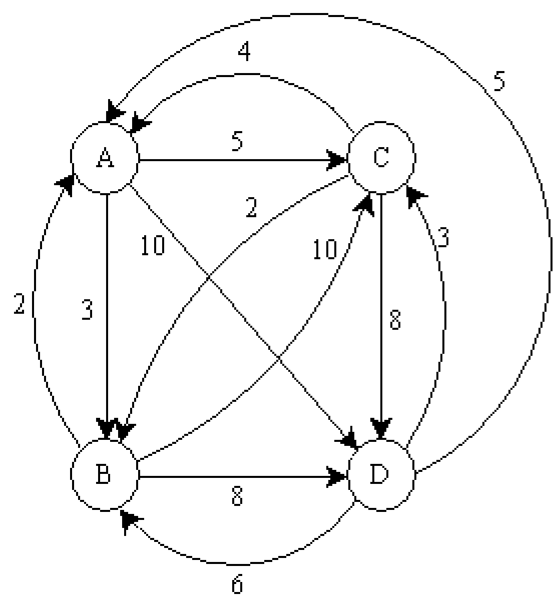
$$= \min\{3+15,\ 5+15,\ 10+7\}$$

$$= \min\{18, 20, 17\}$$

$$= 17$$

Solution:

The graph would be like



Cost(2,∅) = *d* [2,1] = 2

Cost(3,∅) = *d* [3,1] = 4

Cost(4,∅) = *d* [4,1] = 5

Cost(2,{3}) = *d*[2,3] + *Cost*(3,∅)

= 4 + 4 = 8

Cost(2,{4}) = *d*[2,4] + *Cost*(4,∅)

= 8 + 5 = 13

$Cost(3,\{2\}) = d[3,2] + Cost(2,\varnothing)$

$= 2 + 2 = 4$

$Cost(3,\{4\}) = d[3,4] + Cost(4,\varnothing)$

$= 8 + 5 = 13$

$Cost(4,\{2\}) = d[4,2] + Cost(2,\varnothing)$

$= 6 + 2 = 8$

$Cost(4,\{3\}) = d[4,3] + Cost(3,\varnothing)$

$= 3 + 4 = 7$

Now $Cost(2,\{3,4\})$ can be computed as follows:

$$Cost(2,\{3,4\}) = \min\begin{cases} d[2,3] + Cost(3,\{4\}) \\ d[2,4] + Cost(4,\{3\}) \end{cases}$$

$$= \min\begin{cases} 4+13 \\ 8+7 \end{cases}$$

$$= 15$$

$$Cost(3,\{2,4\}) = \min\begin{cases} d[3,2] + Cost(2,\{4\}) \\ d[3,4] + Cost(4,\{2\}) \end{cases}$$

$$= \min\begin{cases} 2+13 \\ 8+8 \end{cases}$$

$$= 15$$

$$Cost(4,\{2,3\}) = \min\begin{cases} d[4,2] + Cost(2,\{3\}) \\ d[4,3] + Cost(3,\{2\}) \end{cases}$$

$$= \min\begin{cases} 6+8 \\ 3+4 \end{cases}$$

$$= 7$$

Finally the $Cost(1,\{2,3,4\})$ can be computed as follows.

$$Cost\left(1,\{2,3,4\}\right) = \min \begin{cases} d[1,2] + Cost\left(2,\{3,4\}\right) \\ d[1,3] + Cost\left(3,\{2,4\}\right) \\ d[1,4] + Cost\left(4,\{2,3\}\right) \end{cases}$$

$$= \min\{3+15,\ 5+15,\ 10+7\}$$

$$= \min\{18, 20, 17\}$$

$$= 17$$

So the minimum cost tour is 17

Path is A $\longrightarrow$ D $\longrightarrow$ C $\longrightarrow$ B $\longrightarrow$ A.

13.11 Solve the following 0–1 Knapsack problem using the dynamic programming approach. Assume that the knapsack capacity is 12.

Items	**Weights**	**Profit**
1	2	14
2	4	20
3	6	24
4	10	30

***Solution*:**

The initial table would be as follows:

	0	1	2	3	4
0	0	0	0	0	0
1	0				
2	0				
3	0				
4	0				

After all items are loaded, the final table would be like.

	0	1	2	3	4
0	0	0	0	0	0
1	0	14	14	14	14
2	0			34	34
3	0				48
4	0				48

Maximum profit is 48

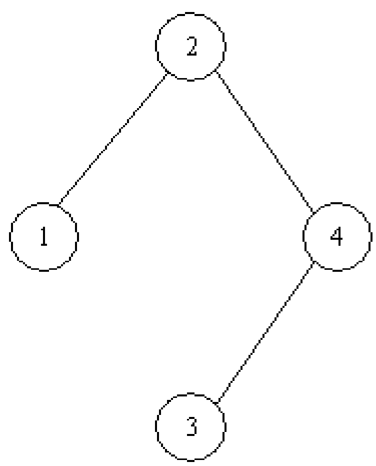
13.12

Find the optimal cost and show the final BST tree for four symbols a_1, a_2, a_3, and a_4 with the following problem

a) $P_1 = \frac{2}{7}, \quad P_2 = \frac{1}{7}, \quad P_3 = \frac{1}{7}, \quad P_4 = \frac{3}{7}$

***Solution*:**

The tree would be



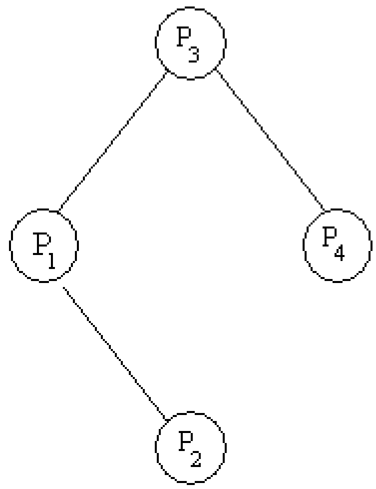
with the Optimal Cost matrix *Optimal Root matrix*

$$\begin{array}{cccc} 2/7 & 4/7 & 7/7 & 14/7 \\ & 1/7 & 3/7 & 8/7 \\ & & 1/7 & 5/7 \\ & & & 3/7 \end{array} \qquad \begin{array}{cccc} 1 & 1 & 1 & 2 \\ & 2 & 2 & 4 \\ & & 3 & 4 \\ & & & 4 \end{array}$$

b) $P_1 = \frac{1}{5}, \quad P_2 = \frac{1}{5}, \quad P_3 = \frac{2}{5}, \quad P_4 = \frac{1}{5}$

Solution:

The tree would be like



with the Cost matrix

$$\begin{matrix} 1/5 & 3/5 & 7/5 & 9/5 \\ & 1/5 & 4/5 & 6/5 \\ & & 2/5 & 4/5 \\ & & & 1/5 \end{matrix}$$

Root matrix

$$\begin{matrix} 1 & 1 & 2 & 3 \\ & 2 & 3 & 3 \\ & & 3 & 3 \\ & & & 4 \end{matrix}$$

13.13 Let there be a single machine with five jobs as shown in the following job and process time table.

Jobs	A	B	C	D	E
Process time	6	9	4	2	3

Find the shortest process time, completion time and mean flow time.

Solution:

Sort the jobs in ascending order to get shortest process time.

D, E, C, A, B.

Completion time is 24.

2, 2+3, 2+3+4, 2+3+4+6, 2+3+4+6+9

2, 5, 9, 15, 24

$$Mean\ \ flow\ time = \frac{2+5+9+15+24}{6}$$

$$= \frac{55}{6} = 9.1\ Hrs.$$

13.14 Let there be five jobs that need to go through machines A(Printing books) and B(Binding machines) in the order AB. The process time of the jobs are as follows.

Jobs	1	2	3	4	5
A	2	3	14	16	20
B	4	3	10	18	24

***Solution*:**

After the 1st job is scheduled

1				

Job	2	3	4	5
A	3	14	16	20
B	3	10	18	24

1	2			

Job	3	4	5
A	14	16	20
B	10	18	24

1	2			3

Job	4	5
A	16	20
B	18	24

1	2	4		3

Job	5
A	20
B	24

1	2	4	5	3

The final elapsed time would be

Job	Machine A Time		Machine B Time		Idle Time	
	In	Out	In	Out	A	B
1	0	2	2	6	-	2
2	2	5	6	9	-	-
4	5	21	21	39	-	12
5	21	37	39	63	-	-
3	37	51	63	73	-	-
					0	14

The elapsed time is 73 – 51 = 22.

13.15 Let there be five jobs that needs to go through machines A and B in the order AB. The processing times of jobs are as follows:

Jobs	1	2	3	4	5
A	2	3	14	16	20
B	4	3	10	18	24

Find the optimal order.

Solution:

The minimum is 2 .

The process queue would be

1				

Jobs	2	3	4	5
A	3	14	16	20
B	3	10	18	24

In case of tie, any job can be chosen as

1	2			

Jobs	3	4	5
A	14	16	20
B	10	18	24

1	2			3

Jobs	4	5
A	16	20
B	18	24

1	2	4		3

Jobs	5
A	20
B	24

1	2	4	5	3

$\therefore$ The optimal order is { 1, 2, 4, 5, 3 }