Chapter 12

12.1Use Gaussian elimination and solve the following problem.

a) $2x_1 + 3x_2 + x_3 = 12$

$x_1 + x_2 + 3x_3 = 10$

$2x_1 + x_2 + x_3 = 8$

***Solution*:**

Add –½times row 1 to row 2

$$\left[\begin{array}{ccc|c} 2 & 3 & 1 & 12 \\ 0 & \frac{-1}{2} & \frac{5}{2} & 10 \\ 2 & 1 & 1 & 8 \end{array}\right]$$

Add –1 times row 1 to row 3

$$\left[\begin{array}{ccc|c} 2 & 3 & 1 & 12 \\ 0 & \frac{-1}{2} & \frac{5}{2} & 4 \\ 0 & -2 & 0 & -4 \end{array}\right]$$

Add -4 times row 2 to row 3

$$\left[\begin{array}{ccc|c} 2 & 3 & 1 & 12 \\ 0 & \frac{-1}{2} & \frac{5}{2} & 4 \\ 0 & 0 & -10 & -20 \end{array}\right]$$

$\therefore x_1 = 2$

$x_2 = 2$

$x_3 = 2$

b)

$2x_1 + 3x_2 + x_3 = 6$

$x_1 + 6x_2 + 7x_3 = 10$

$3x_1 + x_2 + 2x_3 = 8$

***Solution*:**

$$\left[\begin{array}{ccc|c} 2 & 3 & 1 & 6 \\ 1 & 6 & 7 & 14 \\ 3 & 1 & 2 & 5 \end{array}\right]$$

Apply -½ times row 1 to row 2

$$\left[\begin{array}{ccc|c} 2 & 3 & 1 & 6 \\ 0 & \frac{9}{2} & \frac{13}{2} & 11 \\ 3 & 1 & 2 & 5 \end{array}\right]$$

Apply $^{-3}\!/_2$ times row 1 to row 3

$$\left[\begin{array}{ccc|c} 2 & 3 & 1 & 6 \\ 0 & \frac{9}{2} & \frac{13}{2} & 11 \\ 0 & \frac{-7}{2} & \frac{1}{2} & -4 \end{array}\right]$$

Apply $^{7}\!/_9$ times row 2 to row 3

$$\left[\begin{array}{ccc|c} 2 & 3 & 1 & 6 \\ 0 & \frac{9}{2} & \frac{13}{2} & 11 \\ 0 & 0 & \frac{50}{9} & \frac{41}{9} \end{array}\right]$$

$$x_3 = \frac{41}{50}$$

$$x_2 = \frac{63}{50}$$

$$x_1 = \frac{7}{10}$$

∴ Solution is $x_1 = \frac{7}{10}$, $x_2 = \frac{63}{50}$, $x_3 = \frac{41}{50}$.

12.2 Perform LU decomposition of the following matrices using the Gaussion elimination method.

a) $\begin{pmatrix} 2 & 3 & 1 \\ 1 & 1 & 3 \\ 2 & 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 12 \\ 10 \\ 8 \end{pmatrix}$

***Solution*:**

Using the previous problem, one can find

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 0.5 & 1 & 0 \\ 1 & 4 & 1 \end{pmatrix}$$

$$U = \begin{pmatrix} 2 & 3 & 1 \\ 0 & -0.5 & 0.5 \\ 0 & 0 & -2 \end{pmatrix}$$

b) $\begin{pmatrix} 2 & 3 & 1 \\ 1 & 6 & 7 \\ 3 & 1 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 6 \\ 14 \\ 5 \end{pmatrix}$

***Solution*:**

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 0.5 & 1 & 0 \\ 1.5 & -0.77 & 1 \end{pmatrix}$$

$$U = \begin{pmatrix} 2 & 3 & 1 \\ 0 & 4.5 & 6.5 \\ 0 & 0 & 5.55 \end{pmatrix}$$

12.3 Apply count procedure for solving the problem given in exercise 12.2

a) $\begin{pmatrix} 4 & 1 & 1 \\ 2 & 2 & 2 \\ 6 & 4 & 2 \end{pmatrix}$

***Solution*:**

$l_{11} = a_{11} = 4$

$l_{21} = a_{21} = 2$

$l_{31} = a_{31} = 6$

$$u_{12} = \frac{a_{12}}{a_{11}} = \frac{1}{4} = 0.25$$

$$\begin{aligned} l_{22} &= a_{22} - l_{21} u_{12} \\ &= 2 - (2 \times 0.25) = 1.5 \end{aligned}$$

$$\begin{aligned} l_{32} &= a_{32} - l_{31} u_{12} \\ &= 4 - (6 \times 0.25) = 1.9 \end{aligned}$$

$$u_{13} = \frac{a_{13}}{a_{11}} = \frac{1}{4} = 0.25$$

$$\begin{aligned} u_{23} &= \frac{a_{23} - l_{21} \times u_{13}}{l_{22}} \\ &= \frac{3 - 2 \times 0.25}{1.5} = \frac{2.5}{1.5} = 1.66 \end{aligned}$$

$$\begin{aligned} l_{33} &= a_{33} - l_{31} \times u_{13} - l_{32} \times u_{23} \\ &= 2 - 6 \times 0.25 - 1.9 \times 1.66 = 2 - 1.5 - 3.154 = -2.654 \end{aligned}$$

$$\therefore \qquad L = \begin{pmatrix} l_{11} & & \\ l_{21} & l_{22} & \\ l_{31} & l_{32} & l_{33} \end{pmatrix} = \begin{pmatrix} 4 & 0 & 0 \\ 2 & 1.5 & 0 \\ 6 & 1.9 & -2.654 \end{pmatrix}$$

$$U = \begin{pmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0.25 & 0.25 \\ 0 & 1 & 1.66 \\ 0 & 0 & 1 \end{pmatrix}$$

b) $\begin{pmatrix} 4 & 6 & 7 \\ 12 & 3 & 4 \\ 6 & 8 & 7 \end{pmatrix}$

(i) Elements of the 1st column value have the following

$$\begin{pmatrix} 4 \\ 12 \\ 6 \end{pmatrix} \quad l_{11} = 4; l_{21} = 12; l_{31} = 6$$

(ii) Elements of the 1st row

$$u_{12} = \frac{a_{12}}{a_{11}} = \frac{6}{4} = 1.5$$

$$u_{13} = \frac{a_{13}}{a_{11}} = \frac{7}{4} = 1.75$$

$$\therefore \quad matrix \begin{pmatrix} 4 & 1.5 & 1.75 \\ 12 & * & * \\ 6 & * & * \end{pmatrix}$$

(iii) Elements of the 2[nd] column

$$l_{22} = a_{22} - l_{21} \times u_{12} = 8 - 12 \times 1.5 = -15$$
$$l_{32} = a_{32} - l_{31} \times u_{12} = 8 - 6 \times 1.5 \quad = -1$$

$$\therefore \quad matrix \begin{pmatrix} 4 & 1.5 & 1.75 \\ 12 & -15 & * \\ 6 & -1 & * \end{pmatrix}$$

(iv) Elements of the 2[nd] row

$$l_{23} = \frac{a_{23} - l_{21} \times u_{13}}{l_{22}} = \frac{4 - 12 \times 1.75}{-15} = \frac{-17}{-15} = 1.13$$

(v) The last entry

$$u_{33} = a_{33} - l_{33} \times u_{13} - l_{32} \times u_{23}$$
$$= 7 - 6 \times 1.75 - (-1) \times 1.13 = -2.37$$

$$\therefore \quad matrix \begin{pmatrix} 4 & 1.5 & 1.75 \\ 12 & -15 & 1.13 \\ 6 & -1 & -2.37 \end{pmatrix}$$

$$\therefore \quad L = \begin{pmatrix} 4 & 0 & 0 \\ 12 & -15 & 0 \\ 6 & -1 & -2.37 \end{pmatrix}$$

$$K = \begin{pmatrix} 4 & 1.5 & 1.75 \\ 0 & 1 & 1.13 \\ 0 & 0 & 1 \end{pmatrix}$$

12.4 Find matrix inverse for the following matrix using Gaussian elimination method.

a) $\begin{pmatrix} 2 & 4 \\ 3 & 1 \end{pmatrix}$

Multiply row 1 by ½ to get

$$\left(\begin{array}{cc|cc} 1 & 2 & \frac{1}{2} & 0 \\ 3 & 1 & 0 & 1 \end{array}\right)$$

Add -3 times row 1 to row 2

$$\left(\begin{array}{cc|cc} 1 & 2 & \frac{1}{2} & 0 \\ 0 & -5 & \frac{-3}{2} & 1 \end{array}\right)$$

Multiply row 2 by $^{-1}\!/_{5}$

$$\left(\begin{array}{cc|cc} 1 & 2 & \frac{1}{2} & 0 \\ 0 & 1 & \frac{3}{10} & \frac{-1}{5} \end{array}\right)$$

Add -2 times row 2 to row 1 to get

$$\left(\begin{array}{cc|cc} 1 & 0 & \frac{-1}{10} & \frac{2}{5} \\ 0 & 1 & \frac{3}{10} & \frac{-1}{5} \end{array}\right)$$

$\therefore$ Matrix Inverse is $\begin{pmatrix} \frac{-1}{10} & \frac{2}{5} \\ \frac{3}{10} & \frac{-1}{5} \end{pmatrix}$

b) $\begin{pmatrix} 2 & 3 & 1 \\ 1 & 3 & 4 \\ 5 & 6 & 2 \end{pmatrix}$

Solution:

Multiply row 1 by ½ to get

$$\left(\begin{array}{ccc|ccc} 1 & \frac{3}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 3 & 4 & 0 & 1 & 0 \\ 5 & 6 & 2 & 0 & 0 & 1 \end{array}\right)$$

Add -1 times row 1 to row 2

$$\left(\begin{array}{ccc|ccc} 1 & \frac{3}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & \frac{3}{2} & \frac{7}{2} & \frac{-1}{2} & 1 & 0 \\ 5 & 6 & 2 & 0 & 0 & 1 \end{array}\right)$$

Add -5 times row 1 to row 2

$$\left(\begin{array}{ccc|ccc} 1 & \frac{3}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & \frac{3}{2} & \frac{7}{2} & \frac{-1}{2} & 1 & 0 \\ 0 & \frac{-3}{2} & \frac{-1}{2} & \frac{-5}{2} & 0 & 1 \end{array}\right)$$

Multiply row 2 by $^2/_3$

$$\left(\begin{array}{ccc|ccc} 1 & \frac{3}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{7}{3} & \frac{-1}{3} & \frac{2}{3} & 0 \\ 0 & \frac{-3}{2} & \frac{-1}{2} & \frac{-5}{2} & 0 & 1 \end{array}\right)$$

Add $^{-3}/_2$ times row 2 to row 1

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -3 & 1 & -1 & 0 \\ 0 & 1 & \frac{7}{3} & \frac{-1}{3} & \frac{2}{3} & 0 \\ 0 & \frac{-3}{2} & \frac{-1}{2} & \frac{-5}{2} & 0 & 1 \end{array}\right)$$

Add $^3/_2$ times of row 2 to row 3

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -3 & 1 & -1 & 0 \\ 0 & 1 & \frac{7}{3} & \frac{-1}{3} & \frac{2}{3} & 0 \\ 0 & 0 & 3 & -3 & 1 & 1 \end{array}\right)$$

Add 3 times row 3 to row 1

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 0 & 1 \\ 0 & 1 & \frac{7}{3} & \frac{-1}{3} & \frac{2}{3} & 0 \\ 0 & 0 & 1 & -1 & \frac{1}{3} & \frac{1}{3} \end{array}\right)$$

Add –73 times row 3 to row 2

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 0 & 1 \\ 0 & 1 & 0 & 2 & \frac{-1}{9} & \frac{-7}{9} \\ 0 & 0 & 1 & -1 & \frac{1}{3} & \frac{1}{3} \end{array}\right)$$

$$\therefore \text{ Matrix inverse is } \begin{pmatrix} -2 & 0 & 1 \\ 2 & \frac{-1}{9} & \frac{-7}{9} \\ -1 & \frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

12.5 Find the determinant of the following matrices using Crout's LU decomposition.

a) $\begin{pmatrix} 3 & 2 \\ 9 & 6 \end{pmatrix}$

$$\begin{pmatrix} l_{11} & 0 \\ l_{21} & l_{22} \end{pmatrix}\begin{pmatrix} 1 & u_{12} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} l_{11} & l_{11}u_{12}+0 \\ l_{21} & l_{21}u_{12}+l_{22} \end{pmatrix}$$

$\therefore l_{11} = 3 \quad l_{11}u_{12} = 2$

$u_{12} = {}^2\!/_3 = 0.66$

$l_{21} = 4 \qquad l_{21}u_{12} + l_{22}$

$4\times{}^2\!/_3 + l_{22} = 6$

${}^8\!/_3 + l_{22} = 6$

$\therefore l_{22} = 6 - \frac{8}{3} = 3.33$

$\therefore \quad |A| = |L||U|$

$$= \begin{vmatrix} 3 & 0 \\ 3 & 3.33 \end{vmatrix} \begin{vmatrix} 1 & 0.66 \\ 0 & 1 \end{vmatrix}$$

$= 9.99$

b) $\begin{vmatrix} 4 & 6 & 2 \\ 7 & 8 & 2 \\ 1 & 1 & 4 \end{vmatrix}$

The decomposition results in

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 0.571 & 1 & 0 \\ 0.143 & -0.1 & 1.0 \end{pmatrix}$$

$$U = \begin{pmatrix} 7 & 8 & 2 \\ 0 & 1.429 & 0.857 \\ 0 & 0 & 3.8 \end{pmatrix}$$

Therefore, the determinant is 38.01

One can verify this result with the conventional determinant method.

12.6 Let us consider the polynomial

$F(x) = 2 + 4x + 6x^2 + 6x^3$ for $x = 3$,

Use Horner's method to solve the polynomial.

Solution:

Here $a_0 = 2$

$A_1 = 4$

$a_2 = 6$ and $x = 3$

$a_3 = 6$

As per the algorithm, initialize

$S = a_3 = 6$

$i = 1$,

$s = a_{3-1} + s\times x$

$= a_2 + 6\times3 = 6 + 6\times3 = 24$

$i = 2$,

$s = a_{3-2} + s\times x$

$= a_1 + 24\times3 = 4 + 24\times3 = 76$

$i = 3$,

$s = a_{3-3} + s\times x$

$= a_0 + 76\times3 = 2 + 76\times3 = 230$

Since $i = 3$, the algorithm terminates.

Cross check :

$f(3) = 2 + 4\times3 + 6\times9 + 6\times27$

$= 2 + 12 + 54 + 162$

$= 230.$

12.7 Use binary exponentiation method and solve the following functions. Show the intermediate steps.

a) x^{14}

***Solution*:**

The binary representation at $(14)_{10}$ is $(1110)_2$.

Binary digits }	1	1	1	0
Action	Initialize			
Product acc.	x	$x\cdot x^2 = x^3$	$x(x^3)^2 = x^7$	$(x^7)^2 = x^{14}$

$$\therefore x^{14} = \left(x^7\right)^2$$

$$= \left[x\left(x^3\right)^2\right]^2$$

$$= \left[x\left(x\cdot x^2\right)^2\right]^2$$

b) x^{12}

Solution:

The binary representation of

$(12)_{10}$ is 1100

Therefore, the solution is

Binary digits 1 1 0 0

Action Initialize

$xx{\cdot}x^2$ $(x^3)^2$ $(x^6)^2$

Thus, it can be observed that

$$x^{12} = \left(x^6\right)^2$$
$$= \left[\left(x^3\right)^2\right]^2$$
$$= \left[\left(x \square x^2\right)^2\right]^2$$

12.8 Find LCM of the following numbers using GEM

a) 64 and 16

$$LCM\left(m,n\right) = \frac{m \times n}{GCD\left(m,n\right)}$$
$$= \frac{64 \times 16}{GCD\left(64,16\right)} = \frac{1024}{GCD\left(64,16\right)}$$
$$= \frac{1024}{16} = 64$$

b) 128 and 36

$$LCM\left(m,n\right) = \frac{m \times n}{GCD\left(m,n\right)}$$
$$LCM\left(128,36\right) = \frac{128 \times 36}{GCD\left(128,36\right)}$$
$$= \frac{4608}{4} = 1152$$

GCD(128,36) =GCD(36,20) = GCD(20,16) = GCD(16,4) = GCD(4,0) = 4.